- EM pulse (EMP) problems [33],
- EM exploration of minerals [34], and
- EM energy deposition in human bodies [35,36].

It is practically impossible to cover all those applications within the limited scope of this text. In this section, we consider the relatively easier problems of transmission lines and waveguides while the problems of penetration and scattering of EM waves will be treated in the next section. Other applications utilize basically similar techniques.

### 3.7.1 Transmission Lines

The finite difference techniques are suited for computing the characteristic impedance, phase velocity, and attenuation of several transmission lines—polygonal lines, shielded strip lines, coupled strip lines, microstrip lines, coaxial lines, and rectangular lines [12–19]. The knowledge of the basic parameters of these lines is of paramount importance in the design of microwave circuits.

For concreteness, consider the microstrip line shown in Fig. 3.14(a). The geometry in Fig. 3.14(a) is deliberately selected to be able to illustrate how one accounts for discrete inhomogeneities (i.e., homogeneous media separated by interfaces) and lines of symmetry using finite difference technique. The techniques presented are equally applicable to other lines. Due to the fact that the mode is TEM, having components of neither **E** nor **H** fields in the direction of propagation, the fields obey Laplace's equation over the line cross section. The TEM mode assumption provides good approximations if the line dimensions are much smaller than half a wavelength, which means that the operating frequency is far below cutoff frequency for all higher order modes [16]. Also owing to biaxial symmetry about the two axes only one quarter of the cross section need be considered as shown in Fig. 3.14(b).

The finite difference approximation of Laplace's equation, $\nabla^2 V = 0$, has been derived in Eq. (3.26), namely,

$$V(i,j) = \frac{1}{4}\Big[V(i+1,j) + V(i-1,j) + V(i,j+1) + V(i,j-1)\Big]. \tag{3.40}$$

For the sake of conciseness, let us denote

$$\begin{aligned} V_o &= V(i,j) \\ V_1 &= V(i,j+1) \\ V_2 &= V(i-1,j) \\ V_3 &= V(i,j-1) \\ V_4 &= V(i+1,j) \end{aligned} \tag{3.41}$$



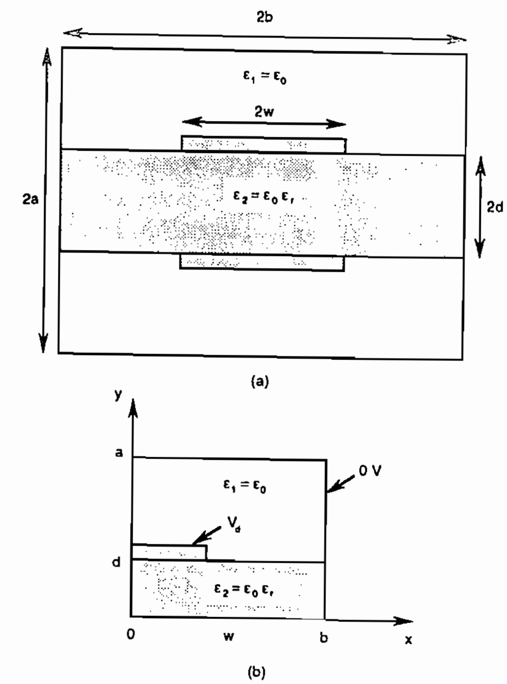


**Figure 3.14** (a) Shielded double strip line with partial dielectric support; (b) problem in (a) simplified by making full use of symmetry.

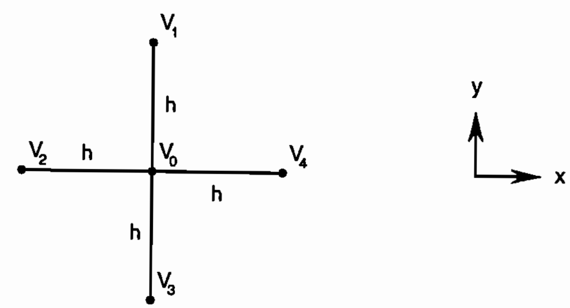


**Figure 3.15** Computation molecule for Laplace's equation.

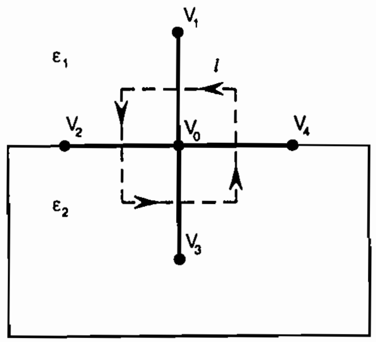


**Figure 3.16** Interface between media of dielectric permittivities $\epsilon_1$ and $\epsilon_2$.

so that Eq. (3.40) becomes

$$\boxed{V_o = \frac{1}{4}\left[V_1 + V_2 + V_3 + V_4\right]} \tag{3.42}$$

with the computation molecule shown in Fig. 3.15. Equation (3.42) is the general formula to be applied to all free nodes in the free space and dielectric region of Fig. 3.14(b).

On the dielectric boundary, the boundary condition,

$$D_{1n} = D_{2n}, \tag{3.43}$$

must be imposed. We recall that this condition is based on Gauss's law for the electric field, i.e.,

$$\oint_\ell \mathbf{D} \cdot d\mathbf{l} = \oint_\ell \epsilon \mathbf{E} \cdot d\mathbf{l} = Q_{\text{enc}} = 0 \tag{3.44}$$

since no free charge is deliberately placed on the dielectric boundary. Substituting $\mathbf{E} = -\nabla V$ in Eq. (3.44) gives

$$0 = \oint_\ell \epsilon \nabla V \cdot d\mathbf{l} = \oint_\ell \epsilon \frac{\partial V}{\partial n} dl \tag{3.45}$$

where $\partial V/\partial n$ denotes the derivative of $V$ normal to the contour $\ell$. Applying Eq. (3.45) to the interface in Fig. 3.16 yields

$$0 = \epsilon_1 \frac{(V_1 - V_0)}{h} h + \epsilon_1 \frac{(V_2 - V_0)}{h}\frac{h}{2} + \epsilon_2 \frac{(V_2 - V_0)}{h}\frac{h}{2}$$
$$+ \epsilon_2 \frac{(V_3 - V_0)}{h} h + \epsilon_2 \frac{(V_4 - V_0)}{h}\frac{h}{2} + \epsilon_1 \frac{(V_4 - V_0)}{h}\frac{h}{2}.$$



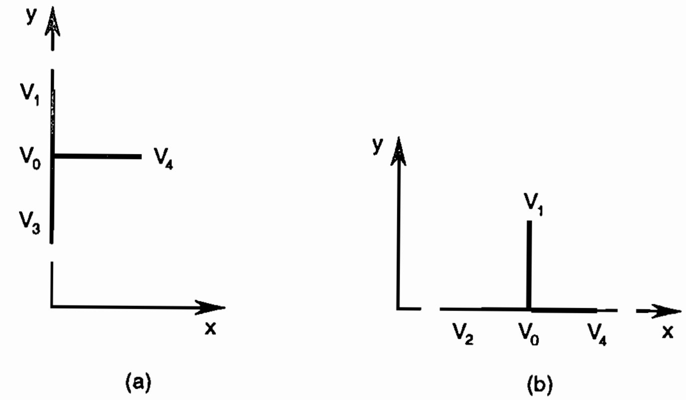


**Figure 3.17** Computation molecule used for satisfying symmetry conditions: (a) $\partial V/\partial x = 0$, (b) $\partial V/\partial y = 0$.

Rearranging the terms,

$$2(\epsilon_1 + \epsilon_2)V_0 = \epsilon_1 V_1 + \epsilon_2 V_3 + \frac{(\epsilon_1 + \epsilon_2)}{2}(V_1 + V_4)$$

or

$$\boxed{V_0 = \frac{\epsilon_1}{2(\epsilon_1 + \epsilon_2)} V_1 + \frac{\epsilon_2}{2(\epsilon_1 + \epsilon_2)} V_3 + \frac{1}{4} V_2 + \frac{1}{4} V_4} \tag{3.46}$$

This is the finite difference equivalent of the boundary condition in Eq. (3.43). Notice that the discrete inhomogeneity does not affect points 2 and 4 on the boundary but affects points 1 and 3 in proportion to their corresponding permittivities. Also note that when $\epsilon_1 = \epsilon_2$, Eq. (3.46) reduces to Eq. (3.42).

On the line of symmetry, we impose the condition

$$\frac{\partial V}{\partial n} = 0. \tag{3.47}$$

This implies that on the line of symmetry along the $y$-axis, ($x = 0$ or $i = 0$) $\dfrac{\partial V}{\partial x} = (V_4 - V_2)/h = 0$ or $V_2 = V_4$ so that Eq. (3.42) becomes

$$\boxed{V_o = \frac{1}{4}\left[V_1 + V_3 + 2V_4\right]} \tag{3.48a}$$

or

$$V(0,j) = \frac{1}{4}\left[V(0,j+1) + V(0,j-1) + 2V(1,j)\right]. \tag{3.48b}$$

On the line of symmetry along the $x$-axis ( $y = 0$ or $j = 0$), $\frac{\partial V}{\partial y} = (V_1 - V_3)/h = 0$ or $V_3 = V_1$ so that

$$\boxed{V_o = \frac{1}{4}\left[2V_1 + V_2 + V_4\right]} \tag{3.49a}$$

or

$$V(i,0) = \frac{1}{4}\left[2V(i,1) + V(i-1,0) + V(i+1,0)\right]. \tag{3.49b}$$

The computation molecules for Eqs. (3.48) and (3.49) are displayed in Fig. 3.17.

By setting the potential at the fixed nodes equal to their prescribed values and applying Eqs. (3.42), (3.46), (3.48), and (3.49) to the free nodes according to the band matrix or iterative methods discussed in Section 3.5, the potential at the free nodes can be determined. Once this is accomplished, the quantities of interest can be calculated.

The characteristic impedance $Z_o$ and phase velocity $u$ of the line are defined as

$$Z_o = \sqrt{\frac{L}{C}} \tag{3.50a}$$

$$u = \frac{1}{\sqrt{LC}} \tag{3.50b}$$

where $L$ and $C$ are the inductance and capacitance per unit length, respectively. If the dielectric medium is nonmagnetic ($\mu = \mu_o$), the characteristic impedance $Z_{oo}$ and phase velocity $u_o$ with the dielectric removed (i.e., the line is air-filled) are given by

$$Z_{oo} = \sqrt{\frac{L}{C_o}} \tag{3.51a}$$

$$u_o = \frac{1}{\sqrt{LC_o}} \tag{3.51b}$$

where $C_o$ is the capacitance per unit length without the dielectric. Combining Eqs. (3.50) and (3.51) yields

$$Z_o = \frac{1}{u_o\sqrt{CC_o}} = \frac{1}{uC} \tag{3.52a}$$

$$u = u_o\sqrt{\frac{C_o}{C}} = \frac{u_o}{\sqrt{\epsilon_{\text{eff}}}} \tag{3.52b}$$

$$\epsilon_{\text{eff}} = \frac{C}{C_o} \tag{3.52c}$$



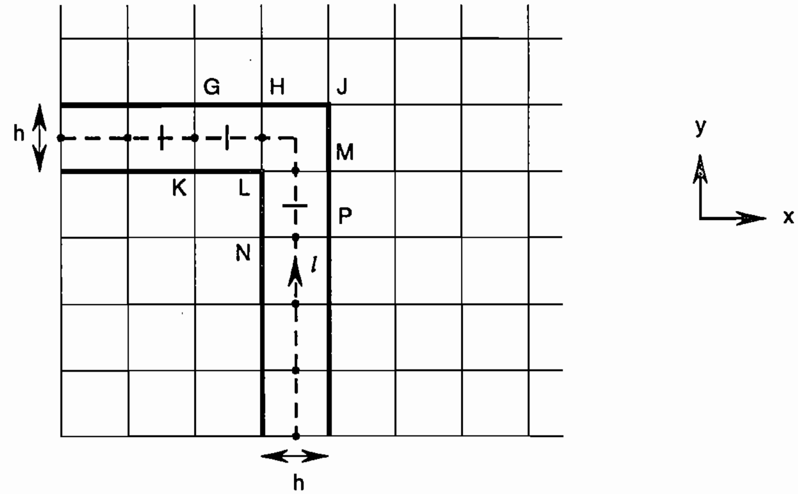


**Figure 3.18** The rectangular path $\ell$ used in calculating charge enclosed.

where $u_o = c = 3 \times 10^8$ m/s, the speed of light in free space, and $\epsilon_{\text{eff}}$ is the effective dielectric constant. Thus to find $Z_o$ and $u$ for an inhomogeneous medium requires calculating the capacitance per unit length of the structure, with and without the dielectric substrate.

If $V_d$ is the potential difference between the inner and the outer conductors,

$$C = \frac{4Q}{V_d}, \tag{3.53}$$

so that the problem is reduced to finding the charge per unit length $Q$. (The factor 4 is needed since we are working on only one quarter of the cross section.) To find $Q$, we apply Gauss's law to a closed path $\ell$ enclosing the inner conductor. We may select $\ell$ as the rectangular path between two adjacent rectangles as shown in Fig. 3.18.

$$Q = \oint_\ell \mathbf{D} \cdot d\mathbf{l} = \oint_\ell \epsilon \frac{\partial V}{\partial n} d\ell$$

$$= \epsilon\left(\frac{V_P - V_N}{\Delta x}\right)\Delta y + \epsilon\left(\frac{V_M - V_L}{\Delta x}\right)\Delta y + \epsilon\left(\frac{V_H - V_L}{\Delta y}\right)\Delta x$$
$$+ \epsilon\left(\frac{V_G - V_K}{\Delta y}\right)\Delta x + \cdots . \tag{3.54}$$

Since $\Delta x = \Delta y = h$,

$$Q = \left(\epsilon V_P + \epsilon V_M + \epsilon V_H + \epsilon V_G + \cdots\right) - \left(\epsilon V_N + 2\epsilon V_L + \epsilon V_K + \cdots\right)$$

or

$$Q = \epsilon_o\Big[\sum \epsilon_{ri} V_i \text{ for nodes } i \text{ on external rectangle GHJMP with corners (such as J) not counted}\Big] - \epsilon_o\Big[\sum \epsilon_{ri} V_i \text{ for nodes } i \text{ on inner rectangle KLN with corners (such as L) counted twice}\Big], \tag{3.55}$$

where $V_i$ and $\epsilon_{ri}$ are the potential and dielectric constant at the $i$th node. If $i$ is on the dielectric interface, $\epsilon_{ri} = (\epsilon_{r1} + \epsilon_{r2})/2$. Also if $i$ is on the line of symmetry, we use $V_i/2$ instead of $V_i$ to avoid including $V_i$ twice in Eq. (3.53), where factor 4 is applied. We also find

$$C_o = 4Q_o/V_d \tag{3.56}$$

where $Q_o$ is obtained by removing the dielectric, finding $V_i$ at the free nodes and then using Eq. (3.55) with $\epsilon_{r1} = 1$ at all nodes. Once $Q$ and $Q_o$ are calculated, we obtain $C$ and $C_o$ from Eqs. (3.53) and (3.56) and $Z_o$ and $u$ from Eq. (3.52).

An outline of the procedure is given below:

(1) Calculate $V$ (with the dielectric space replaced by free space) using Eqs. (3.42), (3.46), (3.48), and (3.49).

(2) Determine $Q$ using Eq. (3.55).

(3) Find $C_o = \dfrac{4Q}{V_d}$.

(4) Repeat steps (1) and (2) (with the dielectric space) and find $C = \dfrac{4Q}{V_d}$.

(5) Finally, calculate $Z_o = \dfrac{1}{c\sqrt{C\,C_o}}$, $c = 3 \times 10^8$ m/s.

The attenuation of the line can be calculated by following similar procedure outlined in [14,20,21]. The procedure for handling boundaries at infinity and that for boundary singularities in finite difference analysis are discussed in [37,38].



### 3.7.2 Waveguides

The solution of waveguide problems is well suited for finite difference schemes because the solution region is closed. This amounts to solving the Helmholtz or wave equation

$$\nabla^2\Phi + k^2\Phi = 0 \tag{3.57}$$

where $\Phi = E_z$ for TM modes or $\Phi = H_z$ for TE modes, while $k$ is the wave number given by

$$k^2 = \omega^2\mu\epsilon - \beta^2. \tag{3.58}$$

The permittivity $\epsilon$ of the dielectric medium can be real for a lossless medium or complex for a lossy medium. We consider all fields to vary with time and axial distance as $\exp j(\omega t - \beta z)$. In the eigenvalue problem of Eq. (3.57), both $k$ and $\Phi$ are to be determined. The cutoff wavelength is $\lambda_c = 2\pi/k_c$. For each value of the cutoff wave number $k_c$, there is a solution for the eigenfunction $\Phi_i$, which represents the field configuration of a propagating mode.

To apply the finite difference method, we discretize the cross section of the waveguide by a suitable square mesh. Applying Eq. (3.24) to Eq. (3.57) gives

$$\boxed{\Phi(i+1,j) + \Phi(i-1,j) + \Phi(i,j+1) + \Phi(i,j-1) - (4 - h^2k^2)\Phi(i,j) = 0} \tag{3.59}$$

where $\Delta x = \Delta y = h$ is the mesh size. Equation (3.59) applies to all the free or interior nodes. At the boundary points, we apply Dirichlet condition ($\Phi = 0$) for the TM modes and Neumann condition ($\partial\Phi/\partial n = 0$) for the TE modes. This implies that at point A in Fig. 3.19, for example,

$$\Phi_A = 0 \tag{3.60}$$

for TM modes. At point A, $\partial\Phi/\partial n = 0$ implies that $\Phi_D = \Phi_E$ so that Eq. (3.57) becomes

$$\Phi_B + \Phi_C + 2\Phi_D - (4 - h^2k^2)\Phi_A = 0 \tag{3.61}$$

for TE modes. By applying Eq. (3.59) and either Eq. (3.60) or (3.61) to all mesh points in the waveguide cross section, we obtain $m$ simultaneous equations involving the $m$ unknowns $(\Phi_1, \Phi_2, \cdots, \Phi_m)$. These simultaneous equations may be conveniently cast into the matrix equation

$$(A - \lambda I)\Phi = 0 \tag{3.62a}$$